

LAPLACIAN COFLOW ON THE 7-DIMENSIONAL HEISENBERG GROUP

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ABSTRACT. We study the Laplacian coflow and the modified Laplacian coflow of G_2 -structures on the 7-dimensional Heisenberg group, showing that even if we start with a coclosed G_2 -structure inducing a nilsoliton, the solution does not necessarily induce a nilsoliton metric. This is in contrast with the Laplacian flow for closed G_2 -structures. Moreover, for the Laplacian coflow we show that the solution is always ancient, that is it is defined in some interval $(-\infty, T)$, with $0 < T < +\infty$. However, for the modified Laplacian coflow, we prove that in some cases the solution is defined only on a finite interval while in other cases the solution is defined for every positive time. Considering the Laplacian coflow and the modified Laplacian coflow on the associated Lie algebra as a bracket flow on \mathbb{R}^7 in a similar way as in [17], we also study the behaviour of the underlying metrics of the solution as t goes to infinity.

1. INTRODUCTION

A 7-dimensional manifold M carries a G_2 -structure if M admits a globally defined 3-form φ , which is called G_2 form, and it can be described locally as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

with respect to some local basis $\{e^1, \dots, e^7\}$ of the local 1-forms on M . Here, e^{127} stands for $e^1 \wedge e^2 \wedge e^7$, and so on. Such a 3-form φ determines a Riemannian metric g_φ and an orientation on M . If ∇ denotes the Levi-Civita connection of g_φ , one can view $\nabla\varphi$ as the torsion of the G_2 -structure φ . Thus, if $\nabla\varphi = 0$, which is equivalent to $d\varphi = 0$ and $d\star_\varphi\varphi = 0$, where \star_φ is the Hodge star operator with respect to g_φ , one says that the G_2 -structure is torsion-free.

The different classes of G_2 -structures can be described in terms of the exterior derivatives $d\varphi$ and $d\star_\varphi\varphi$ [2, 7]. If $d\varphi = 0$, then the G_2 -structure is called *closed* (or *calibrated* in the sense of Harvey and Lawson [10]) and if φ is coclosed, that is if $\star_\varphi\varphi$ is closed, then the G_2 -structure is called *coclosed* (or *cocalibrated* [10]).

Since Hamilton introduced the Ricci flow in 1982 [9], geometric flows have been an important tool in studying geometric structures on manifolds. The Laplacian flow for

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closed G_2 -structures on a 7-manifold M has been introduced by Bryant in [2], and it is given by

$$\begin{cases} \frac{\partial}{\partial t} \varphi(t) = \Delta_t \varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi, \end{cases}$$

where $\varphi(t)$ is a closed G_2 form on M , $\Delta_t = d d^* + d^* d$ is the Hodge Laplacian operator associated with the metric $g_{\varphi(t)}$ induced by the 3-form $\varphi(t)$, and φ is the initial closed G_2 -structure. A short-time existence and uniqueness for this flow, in the case of compact manifolds, has been proved in [3]. Regarding the long-time behavior of the Laplacian flow on compact manifolds M , Lotay and Wei in [20] have proved recently that if the initial closed G_2 form φ is such that its torsion is sufficiently small (in a suitable sense), then the Laplacian flow of φ will exist for all time and converge to a torsion-free G_2 -structure. Non-compact examples where the flow converges to a flat G_2 -structure have been given in [6].

A Shi-type derivative estimates for the Riemann curvature tensor and torsion tensor along the Laplacian flow has been determined in [19], and in [21] it is proved that for each fixed positive time $t \in (0, T]$, $(M, \varphi(t), g_{\varphi(t)})$ is real analytic. Consequently, any Laplacian soliton is real analytic. Moreover, solitons of the Laplacian flow of G_2 -structures in the homogeneous case have been studied recently by Lauret in [18] using the bracket flow and the algebraic soliton approach.

Some work has also been done on other related flows of G_2 -structures - such as the *Laplacian coflow*, or *flow*, for coclosed G_2 -structures. This coflow has been originally proposed by Karigiannis, McKay and Tsui in [15] and, for an initial coclosed G_2 form φ with $\psi = \star_{\varphi} \varphi$, it is given by

$$(1) \quad \frac{\partial}{\partial t} \psi(t) = -\Delta_t \psi(t), \quad d\psi(t) = 0, \quad \psi(0) = \psi,$$

where $\psi(t)$ is the Hodge dual 4-form of a G_2 -structure $\varphi(t)$, that is $\psi(t) = \star_t \varphi(t)$, Δ_t is the Hodge Laplacian operator with respect to the Riemannian metric $g_{\varphi(t)}$. This flow preserves the condition of the G_2 -structure being coclosed, that is $\psi(t)$ is closed for any t , and it was studied in [15] for two explicit examples of coclosed G_2 -structures with symmetry, namely for warped products of an interval, or a circle, with a compact 6-manifold N which is taken to be either a nearly Kähler manifold or a Calabi-Yau manifold. Nevertheless, in [8] it was shown that the coflow (1) is not even a weakly parabolic flow, and that the symbol of the operator Δ_t , acting on 4-forms, has a mixed signature. But no general result is known about the short time existence of the coflow (1).

A *modified Laplacian coflow* was introduced by Grigorian in [8]

$$(2) \quad \frac{\partial}{\partial t} \psi(t) = \Delta_t \psi(t) + 2d\left((A - \text{Tr}_t(\tau(t)))\varphi(t)\right), \quad d\psi(t) = 0, \quad \psi(0) = \psi,$$

where $\text{Tr}_t(\tau(t))$ is the trace of the full torsion tensor $\tau(t)$ of the G_2 -structure defined by $\varphi(t)$, and A is a fixed positive constant (see Section 3 for the details). Moreover, in [8] it is proved that the coflow (2) is weakly parabolic in the direction of closed forms $\psi(t)$ up to diffeomorphisms and, on compact manifolds, it has a unique solution $\psi(t)$ for the short time period $t \in [0, \epsilon]$, for some $\epsilon > 0$.

In this paper, we study the coflows (1) and (2) in the case of the 7-dimensional Heisenberg group H with the coclosed G_2 -structures given in Section 2. In particular, we show a coclosed G_2 form φ on H inducing a nilsoliton metric but neither the solution of the coflow (1) nor the solution of the coflow (2) for φ determine a nilsoliton on H (Theorem 5, Theorem 8). This is in contrast with the Laplacian flow for closed G_2 -structures [6].

As we mentioned before, it is not known any general result on the short time existence of solution for the coflow (1). Nevertheless, in Theorem 4 and Theorem 5, we show that the solution of the coflow (1) for each of the one of the coclosed G_2 -structures considered on the Heisenberg group is always *ancient*, that is it is defined on a time interval of the form $(-\infty, T)$, where $T > 0$ is a real number. To our knowledge, these are the first examples of non-compact manifolds having a coclosed G_2 -structure with long time existence of solution for (1). However, we prove that the solution of the coflow (2) for some coclosed G_2 forms on H is defined only on a finite interval (Theorem 8) and, for other coclosed G_2 forms, the coflow (2) exists for all time (Theorem 6, part ii) or it is ancient (Theorem 6, part i), and Theorem 7).

Moreover, considering the coflows (1) and (2) on the associated Lie algebra as a bracket flow on \mathbb{R}^7 , in a similar way as Lauret did in [17] for the Ricci flow, we show that the underlying metrics $g(t)$ of the solution in Theorems 4 and 7 converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, as t goes to infinity. Indeed, by [17, Proposition 2.8] the convergence of the metrics in the C^∞ uniformly on compact sets in \mathbb{R}^7 is equivalent to the convergence of the nilpotent Lie brackets $\mu(t)$ in the algebraic subset of nilpotent Lie brackets $\mathcal{N} \subset (\Lambda^2 \mathbb{R}^7)^* \otimes \mathbb{R}^7$ with the usual vector space topology.

2. COCLOSED G_2 -STRUCTURES ON THE HEISENBERG GROUP

In this section, we show some coclosed G_2 -structures on the 7-dimensional Heisenberg group H , and we prove that each of them induces the nilsoliton on H determined in [5] (see [4, 13, 16] for details on nilsolitons).

A 7-dimensional manifold M is said to admit a G_2 -structure if there is a reduction of the structure group of its frame bundle from $\text{GL}(7, \mathbb{R})$ to the exceptional Lie group G_2 , which can actually be viewed naturally as a subgroup of $\text{SO}(7)$. Thus, a G_2 -structure determines a Riemannian metric and an orientation on M . In fact, one can prove that the presence of a G_2 -structure is equivalent to the existence of a differential

3-form φ (the G_2 form) on M , which induces the Riemannian metric g_φ given by

$$(3) \quad g_\varphi(X, Y) \text{ vol} = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi,$$

for any vector fields X, Y on M , where vol is the volume form on M , and ι_X denotes the contraction by X . Let \star_φ be the Hodge star operator determined by g_φ and the orientation induced by φ . We will always write ψ to denote the dual 4-form of a G_2 -structure φ , that is

$$\psi = \star_\varphi \varphi.$$

A manifold M has a *coclosed* (or *cocalibrated*) G_2 -structure if there is a G_2 -structure on M such that the G_2 form φ is coclosed, that is $d\psi = 0$.

Now, let G be a 7-dimensional simply connected nilpotent Lie group with Lie algebra \mathfrak{g} . Then, a G_2 -structure on G is *left invariant* if and only if the corresponding 3-form φ is left invariant. Thus, a left invariant G_2 -structure on G corresponds to an element φ of $\Lambda^3(\mathfrak{g}^*)$ that can be written as

$$(4) \quad \varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

with respect to some orthonormal coframe $\{e^1, \dots, e^7\}$ of the dual space \mathfrak{g}^* , where e^{127} stands for $e^1 \wedge e^2 \wedge e^7$, and so on. So the dual form $\psi = \star_\varphi \varphi$ has the following expression

$$(5) \quad \psi = e^{1234} + e^{1256} + e^{1367} + e^{1457} + e^{2357} - e^{2467} + e^{3456}.$$

Note that in order to recover the left invariant G_2 form φ from the 4-form $\star_\varphi \varphi$ we need to fix an orientation of \mathfrak{g} . In fact, the stabilizer of $\star_\varphi \varphi$ in $\text{GL}(7, \mathbb{R})$ is $G_2 \times \mathbb{Z}_2$ since the matrix $-Id$ preserves the form $\star_\varphi \varphi$, and so the latter fails to determine the overall orientation.

By [22], the unique simply connected nilpotent Lie groups admitting left invariant Einstein metrics are Abelian Lie groups. Natural generalizations of left invariant Einstein metrics are *nilsolitons*, which have been introduced by Lauret in [18]. Recall that a left invariant metric g on a nilpotent Lie group (of arbitrary dimension) is called *nilsoliton* if its Ricci tensor satisfies the condition

$$(6) \quad \text{Ric}(g) = \lambda I + D,$$

where λ is a real number, and D is a derivation of the corresponding Lie algebra. Not all nilpotent Lie groups admit nilsoliton metrics, but if a nilsoliton exists, then it is unique up to automorphism and scaling [17].

Recall that the seven dimensional Heisenberg group H is the simply connected nilpotent Lie group whose Lie algebra \mathfrak{h} is defined by

$$(7) \quad \mathfrak{h} = \left(0, 0, 0, 0, 0, 0, \frac{\sqrt{6}}{6}(e^{12} + e^{34} + e^{56}) \right).$$

This notation means that the dual space \mathfrak{h}^* is spanned by $\{e^1, \dots, e^7\}$ satisfying

$$de^i = 0, \quad 1 \leq i \leq 6, \quad de^7 = \frac{\sqrt{6}}{6}(e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6).$$

By [5] the metric

$$(8) \quad g = \sum_{i=1}^7 (e^i)^2$$

is a *nilsoliton* on H because its Ricci tensor

$$Ric(g) = \text{diag}\left(-\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{4}\right)$$

satisfies (6), for $\lambda = -\frac{1}{12}$ and

$$D = \text{diag}\left(0, 0, 0, 0, 0, 0, \frac{1}{3}\right).$$

Proposition 1. *There are coclosed G_2 -structures on the Heisenberg group H inducing the nilsoliton metric (8).*

Proof. Let us consider the left invariant 3-form φ_1 on H defined by

$$(9) \quad \varphi_1 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

According with (4), φ_1 defines a G_2 -structure on H , and $\{e^1, \dots, e^7\}$ is the corresponding orthonormal coframe of \mathfrak{h}^* . So the metric g_{φ_1} , given by (3), is the nilsoliton (8), that is $g_{\varphi_1} = g$.

Let \star be the Hodge star operator determined by g . Then, by (5) the Hodge dual form $\star\varphi_1$ has the following expression

$$\star\varphi_1 = e^{1234} + e^{1256} + e^{1367} + e^{1457} + e^{2357} - e^{2467} + e^{3456}.$$

Using (7), we see that $\star\varphi_1$ is closed since each term on the right side of the last equality is closed. So φ_1 defines a coclosed G_2 -structure on H .

Now, we define the 3-form φ_2 on H by

$$(10) \quad \varphi_2 = e^{127} - e^{347} - e^{567} + e^{135} - e^{146} + e^{236} + e^{245}.$$

By (4), φ_2 defines a left invariant G_2 -structure on H , and $\{e^1, e^2, -e^3, e^4, -e^5, e^6, e^7\}$ is the corresponding orthonormal coframe of \mathfrak{h}^* . Thus, $g_{\varphi_2} = g$, that is φ_2 induces also the nilsoliton (8). Moreover,

$$\star\varphi_2 = -e^{1234} - e^{1256} - e^{1367} - e^{1457} + e^{2357} - e^{2467} + e^{3456}.$$

Using again (7), we have that $\star\varphi_2$ is closed or, equivalently, φ_2 defines a coclosed G_2 -structure on H . \square

Remark 1. Note that if φ is a coclosed G_2 form on \mathfrak{h} inducing the orientation $\{e^1, \dots, e^7\}$ and the nilsoliton (8), then $-\varphi$ is also a coclosed G_2 form on \mathfrak{h} inducing the nilsoliton (8). Moreover, the coflow (1) for $-\varphi$ and the coflow (1) for φ are both the same, since the volume form defined by the G_2 form $-\varphi$ is the opposite to the one defined by φ , and $\star_\varphi \varphi = \star_{-\varphi} -\varphi$, where \star_φ and $\star_{-\varphi}$ denote respectively the Hodge star operator induced by φ and $-\varphi$. (On the coflow (2) for φ and for $-\varphi$, see Remark 2.)

3. ON THE COFLOWS OF COCOCLOSED G_2 -STRUCTURES

Here we show the expression of each one of the coflows (1) and (2) in terms of the intrinsic torsion forms of a coclosed G_2 -structure [2, 8].

Let M be a 7-dimensional manifold with a G_2 -structure defined by a 3-form φ . Denote by ψ the 4-form $\psi = \star_\varphi \varphi$, where \star_φ is the Hodge star operator of the metric g_φ induced by φ . Let $(\Omega^*(M), d)$ be the de Rham complex of differential forms on M . Then, Bryant in [2] proved that the forms $d\varphi$ and $d\psi$ are such that

$$(11) \quad \begin{cases} d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + \star_\varphi \tau_3, \\ d\psi = 4\tau_1 \wedge \psi - \star_\varphi \tau_2, \end{cases}$$

where $\tau_0 \in \Omega^0(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega_{14}^2(M)$ and $\tau_3 \in \Omega_{27}^3(M)$. Here $\Omega_{14}^2(M)$ and $\Omega_{27}^3(M)$ are the spaces

$$\Omega_{14}^2(M) = \{\alpha \in \Omega^2(M) \mid \alpha \wedge \varphi = -\star_\varphi \alpha\},$$

$$\Omega_{27}^3(M) = \{\beta \in \Omega^3(M) \mid \beta \wedge \varphi = 0 = \beta \wedge \star_\varphi \varphi\}.$$

The differential forms τ_i ($i = 0, 1, 2, 3$) that appear in (11), are called the *intrinsic torsion forms* of φ . According to Grigorian [8] the *full torsion tensor* τ of φ is the tensor field on M given by

$$\tau = \frac{1}{4}\tau_0 g_\varphi - \iota_{\tau_1} \varphi - \frac{1}{3}j_\varphi(\tau_3) + \frac{1}{2}\tau_2,$$

where ι_{τ_1} denotes the contraction by τ_1 using the metric g_φ induced by φ (that is, if U is the vector field on M such that $\tau_1 = \iota_U g$, then $\iota_{\tau_1} \varphi = \iota_U \varphi$) and $j_\varphi : \Omega^3(M) \rightarrow S^2(M)$ is the map defined by

$$j_\varphi(\gamma)(X, Y) = \star_\varphi \left((\iota_X \varphi) \wedge (\iota_Y \varphi) \wedge \gamma \right),$$

where $\gamma \in \Omega^3(M)$, and X, Y are vector fields on M [2]. In particular, by [2] j_φ is an isomorphism between the space $\Omega_{27}^3(M)$ and the space $S_0^2(M)$ of trace-free symmetric 2-tensors on M .

Recall that φ defines a coclosed G_2 -structure on M if ψ is closed, that is $d\psi = 0$. In this case, (11) implies that the forms τ_1 and τ_2 vanish, and so the full torsion tensor τ has the following expression

$$\tau = \frac{1}{4}\tau_0 g_\varphi - \frac{1}{3}j_\varphi(\tau_3).$$

Since $\tau_3 \in \Omega_{27}^3(M)$, the trace of $j_\varphi(\tau_3)$ vanishes. Therefore, $\text{Tr}(\tau)$ of τ is given by

$$(12) \quad \text{Tr}(\tau) = \frac{1}{4}\tau_0 \text{Tr}(g_\varphi) = \frac{7}{4}\tau_0.$$

Lemma 2. *Let M be a 7-dimensional manifold with a coclosed G_2 form φ . Denote by τ_0 and τ_3 the torsion forms of φ . Then, the torsion forms $\tilde{\tau}_0$ and $\tilde{\tau}_3$ of $-\varphi$ satisfy*

$$(13) \quad \tilde{\tau}_0 = -\tau_0, \quad \tilde{\tau}_3 = \tau_3.$$

Proof. Using (11), we see that $\tilde{\tau}_0 = -\tau_0$ and $\tilde{\tau}_3 = \tau_3$ since $\star_{-\varphi} = -\star_\varphi$. \square

Proposition 3. *Let M be a 7-dimensional manifold with a coclosed G_2 form φ . Then, the coflow (1) for φ has the following expression*

$$(C) \quad \begin{aligned} \frac{\partial}{\partial t}\psi(t) &= -d(\tau_0(t)) \wedge \varphi(t) - (\tau_0(t))^2 \psi(t) - \tau_0(t) \star_t \tau_3(t) - d\tau_3(t), \\ d\psi(t) &= 0, \quad \varphi(0) = \varphi, \end{aligned}$$

and the coflow (2) is expressed as

$$(G) \quad \begin{aligned} \frac{\partial}{\partial t}\psi(t) &= \tau_0(t) \left(2A - \frac{5}{2}\tau_0(t) \right) \psi(t) + \left(2A - \frac{5}{2}\tau_0(t) \right) \star_t \tau_3(t) + d\tau_3(t) \\ &\quad + \frac{5}{2}\varphi(t) \wedge d\tau_0(t), \\ d\psi(t) &= 0, \quad \varphi(0) = \varphi, \end{aligned}$$

where $\tau_0(t)$ and $\tau_3(t)$ are the torsion forms of $\varphi(t)$ (according with (11)), \star_t is the Hodge star operator with respect to the Riemannian metric $g_{\varphi(t)}$ induced by $\varphi(t)$ and A is a fixed positive constant.

Proof. The coflows (1) and (2) preserve both the closeness of $\psi(t) = \star_t \varphi(t)$. Therefore, by (11) and the vanishing of the torsion forms $\tau_1(t)$ and $\tau_2(t)$ of $\varphi(t)$,

$$d\varphi(t) = \tau_0(t)\psi(t) + \star_t \tau_3(t).$$

Hence,

$$\begin{aligned} \Delta_t \psi(t) &= dd^* \psi(t) = d \star_t d\varphi(t) = d \star_t \left(\tau_0(t)\psi(t) + \star_t \tau_3(t) \right) \\ &= d(\tau_0(t)) \wedge \varphi(t) + \tau_0(t)^2 \psi(t) + \tau_0(t) \star_t \tau_3(t) + d\tau_3(t), \end{aligned}$$

and

$$\begin{aligned} 2d\left((A - \text{Tr}(\tau(t)))\varphi(t)\right) &= 2d\left((A - \frac{7}{4}\tau_0(t))\varphi(t)\right) \\ &= -\frac{7}{2}d(\tau_0(t)) \wedge \varphi(t) + \left(2A - \frac{7}{2}\tau_0(t)\right) (\tau_0(t)\psi(t) + \star_t \tau_3(t)). \end{aligned}$$

Thus,

$$\begin{aligned} \Delta_t \psi + 2d\left((A - \text{Tr}(\tau(t)))\varphi(t)\right) &= -\frac{5}{2}d(\tau_0(t)) \wedge \varphi(t) + \tau_0(t) \left(2A - \frac{5}{2}\tau_0(t)\right) \psi(t) \\ &\quad + \left(2A - \frac{5}{2}\tau_0(t)\right) \star_t \tau_3(t) + d\tau_3(t), \end{aligned}$$

and the Proposition follows. \square

Remark 2. Note that (13) and Proposition 3 imply that the solution of the coflow (G) for φ (if such a solution exists) changes when the initial coclosed G_2 form is $-\varphi$ instead of φ (see Theorem 6 and Theorem 7). However, the study of the coflow (C) is independent of whether the initial condition is φ or $-\varphi$.

Remark 3. By [8], since $\text{Tr}(\tau(t)) = \frac{7}{4}\tau_0(t)$, as long as the condition $0 \leq \frac{7}{4}\tau_0(t) \leq \frac{4}{3}A$ holds for the time of existence, we have the following inequality for the volume

$$A \int_M \frac{7}{4} \tau_0(t) \text{vol} \geq \int_M \frac{3}{4} \left(\frac{7}{4} \tau_0(t)\right)^2 \text{vol}.$$

4. EXPLICIT SOLUTIONS FOR THE LAPLACIAN COFLOW

We solve the coflow (1) on the 7-dimensional Heisenberg group when the initial coclosed G_2 form is equal to φ_i ($i = 1, 2$) where φ_1 and φ_2 are defined by (9) and (10), respectively. In each of these cases, we show that the solution is ancient. Moreover, we prove that the solution of the coflow (1) for φ_1 determines a nilsoliton for the whole time of existence, but this does not happen with the solution of (1) for φ_2 .

Theorem 4. *Let H be the seven dimensional Heisenberg group whose Lie algebra is defined by (7). Then, the solution of the Laplacian coflow (1) with the initial coclosed G_2 form φ_1 , defined by (9), is given by*

$$(14) \quad \varphi(t) = \frac{1}{y(t)} (e^{127} + e^{347} + e^{567}) + y(t)^3 (e^{135} - e^{146} - e^{236} - e^{245}), \quad t \in \left(-\infty, \frac{3}{5}\right),$$

where $y = y(t)$ is the positive function

$$(15) \quad y(t) = \sqrt[10]{1 - \frac{5}{3}t}.$$

The underlying metrics g_t of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in H as t goes to $-\infty$. Moreover, $\varphi(t)$ induces a nilsoliton for every time.

Proof. For each $t \in (-\infty, \frac{3}{5})$, we consider the basis $\{f^1(t), \dots, f^7(t)\}$ of left invariant 1-forms on H defined by

$$\begin{aligned} f^i &= f^i(t) = y(t) e^i, & 1 \leq i \leq 6, \\ f^7 &= f^7(t) = y(t)^{-3} e^7, \end{aligned}$$

where the function $y = y(t)$ is given by (15). Then, $f^i(0) = e^i$, for $i \in \{1, \dots, 7\}$, and the structure equations of H , with respect to the basis $\{f^1(t), \dots, f^7(t)\}$, are

$$(16) \quad df^i = 0, \quad 1 \leq i \leq 6, \quad df^7 = \frac{\sqrt{6}}{6} y(t)^{-5} (f^{12} + f^{34} + f^{56}).$$

Now, for any t , the 3-form $\varphi(t)$ defined by (14) has the following expression

$$(17) \quad \varphi(t) = f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}.$$

Note that $\varphi(0) = \varphi_1$ and, for any t , the 3-form $\varphi(t)$ on H induces the metric g_t such that the coframe $\{f^1(t), \dots, f^7(t)\}$ of \mathfrak{h}^* is orthonormal. Denote by \star_t the Hodge star operator determined by g_t . Using (4), (5) and (16), we have $d \star_t \varphi(t) = 0$, where the 4-form

$$\star_t \varphi(t) = f^{1234} + f^{1256} + f^{1367} + f^{1457} + f^{2357} - f^{2467} + f^{3456}.$$

So, in terms of the coframe $\{e^1, \dots, e^7\}$ of \mathfrak{h}^* , $\star_t \varphi(t)$ has the following expression

$$\star_t \varphi(t) = y(t)^4 (e^{1234} + e^{1256} + e^{3456}) + e^{1367} + e^{1457} + e^{2357} - e^{2467}.$$

Thus,

$$(18) \quad \frac{d}{dt} (\star_t \varphi(t)) = 4y(t)^3 y'(t) (e^{1234} + e^{1256} + e^{3456}).$$

Moreover, using (16) and (17), we have

$$\begin{aligned} -\Delta_t \star_t \varphi(t) &= -d \star_t d\varphi(t) = -\frac{\sqrt{6}}{3} y(t)^{-5} d \star_t (f^{1234} + f^{1256} + f^{3456}) \\ &= -\frac{2}{3} y(t)^{-10} (f^{1234} + f^{1256} + f^{3456}), \end{aligned}$$

or, equivalently,

$$-\Delta_t \star_t \varphi(t) = -\frac{2}{3} y(t)^{-6} (e^{1234} + e^{1256} + e^{3456}).$$

The last equality and (18) prove that (14) is the solution of the coflow (1) when the function $y = y(t)$ is given by (15).

We study the behavior of the underlying metric g_t of the solution in the limit for $t \rightarrow -\infty$. If we think of the coflow (1) as a one parameter family of G_2 manifolds with a coclosed G_2 -structure, it can be checked that, in the limit, the resulting manifold has vanishing curvature. In fact, we know that the metric g_t determined by $\varphi(t)$ is such that $g_t = \sum_{i=1}^7 (f^i(t))^2$. Take the orthonormal basis $\{f_1(t), \dots, f_7(t)\}$ of \mathfrak{h} dual

to the basis $\{f^1(t), \dots, f^7(t)\}$. Then, taking into account the symmetry properties of the Riemannian curvature $R(g_t)$, we obtain that the non-vanishing components of the Riemannian curvature, with respect to the basis $\{f_1(t), \dots, f_7(t)\}$, are given by

$$\begin{aligned} R_{1212} &= (1/8) y(t)^{-10}, & R_{1234} &= (1/12) y(t)^{-10}, & R_{1256} &= (1/12) y(t)^{-10}, \\ R_{1717} &= -(1/24) y(t)^{-10}, & R_{2727} &= -(1/24) y(t)^{-10}, & R_{3434} &= (1/8) y(t)^{-10}, \\ R_{3456} &= (1/12) y(t)^{-10}, & R_{3737} &= -(1/24) y(t)^{-10}, & R_{4747} &= -(1/24) y(t)^{-10}, \\ R_{5656} &= (1/8) y(t)^{-10}, & R_{5757} &= -(1/24) y(t)^{-10}, & R_{6767} &= -(1/24) y(t)^{-10}. \end{aligned}$$

Therefore,

$$\|R(g_t)\|_{g_t}^2 = \frac{23}{48} y(t)^{-20},$$

and hence $\lim_{t \rightarrow -\infty} R(g_t) = 0$.

Moreover, one can check that the Ricci tensor $\text{Ric}(g_t)$, with respect to the basis $\{f_1(t), \dots, f_7(t)\}$, is given by

$$\text{Ric}(g_t) = y(t)^{-10} \text{diag} \left(-\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{4} \right).$$

Thus,

$$\text{Ric}(g_t) = \lambda_t I + D_t,$$

where $\lambda_t = -\frac{1}{12y(t)^{10}}$, and

$$D_t = \text{diag} \left(0, 0, 0, 0, 0, 0, \frac{1}{3} y(t)^{-10} \right)$$

is a derivation of \mathfrak{h} , for every fixed t . So, g_t is a nilsoliton on H , for every t . \square

Theorem 5. *The solution of the Laplacian coflow (1) with initial coclosed G_2 form φ_2 , defined by (10), is ancient and it is given by*

$$(19) \quad \varphi(t) = \frac{y(t)}{z(t)^2} e^{127} - \frac{1}{y(t)} e^{347} - \frac{1}{y(t)} e^{567} + y(t) z(t)^2 (e^{135} - e^{146} + e^{236} + e^{245}),$$

where the functions $y = y(t)$ and $z = z(t)$ satisfy

$$(20) \quad \begin{cases} \frac{d}{dt} y(t) = -\frac{1}{12} \frac{y(t)^4 + z(t)^4}{y(t)^5 z(t)^8}, & \frac{d}{dt} z(t) = \frac{1}{12} \frac{z(t)^2 - y(t)^2}{y(t)^4 z(t)^7}, \\ y(0) = 1, & z(0) = 1. \end{cases}$$

Moreover, the unique G_2 form $\varphi(t)$ inducing a nilsoliton is the one for $t = 0$.

Proof. By Cauchy Theorem, there exist a maximal open interval I , containing 0, and two smooth functions $y, z : I \rightarrow (0, +\infty)$, which are the unique solution of (20).

To prove that (19) is the solution to the coflow (1) for φ_2 we proceed as follows. For each $t \in I$, we consider the basis $\{f^1(t), \dots, f^7(t)\}$ of left invariant 1-forms on H defined by

$$\begin{aligned} f^i &= f^i(t) = y(t) e^i, & i &= 1, 2, \\ f^i &= f^i(t) = z(t) e^i, & i &= 3, \dots, 6, \\ f^7 &= f^7(t) = y(t)^{-1} z(t)^{-2} e^7, \end{aligned}$$

where the functions $y = y(t)$ and $z = z(t)$ satisfy (20). Then, $f^i(0) = e^i$, for $i \in \{1, \dots, 7\}$, and the structure equations of H , with respect to the basis $\{f^1(t), \dots, f^7(t)\}$, are

$$(21) \quad df^i = 0, \quad 1 \leq i \leq 6, \quad df^7 = \frac{\sqrt{6}}{6} y(t)^{-1} z(t)^{-2} \left(y(t)^{-2} f^{12} + z(t)^{-2} f^{34} + z(t)^{-2} f^{56} \right).$$

Moreover, for any $t \in I$, the 3-form $\varphi(t)$ defined by (19) has the following expression

$$(22) \quad \varphi(t) = f^{127} - f^{347} - f^{567} + f^{135} - f^{146} + f^{236} + f^{245}.$$

Note that $\varphi(0) = \varphi_2$ and, for any $t \in I$, the 3-form $\varphi(t)$ on H induces the metric g_t such that $\{f^1(t), \dots, f^7(t)\}$ is an orthonormal basis of \mathfrak{h}^* . Denote by \star_t the Hodge star operator determined by g_t . Using (4), (5) and (21), we have $d(\star_t \varphi(t)) = 0$, where $\star_t \varphi(t)$ is given by

$$\star_t \varphi(t) = -f^{1234} - f^{1256} - f^{1367} - f^{1457} + f^{2357} - f^{2467} + f^{3456}.$$

Thus, in terms of the coframe $\{e^1, \dots, e^7\}$ of \mathfrak{h}^* , the 4-form $\star_t \varphi(t)$ has the following expression

$$\star_t \varphi(t) = y(t)^2 z(t)^2 (-e^{1234} - e^{1256}) - e^{1367} - e^{1457} + e^{2357} - e^{2467} + z(t)^4 e^{3456}.$$

Hence,

$$\frac{d}{dt} (\star_t \varphi(t)) = 2 \left(y(t) z(t)^2 y'(t) + y(t)^2 z(t) z'(t) \right) (-e^{1234} - e^{1256}) + 4 z(t)^3 z'(t) e^{3456}.$$

On the other hand, using (21) and (22), we have

$$\begin{aligned} -\Delta_t \star_t \varphi(t) &= -d \star_t d\varphi(t) = -\frac{\sqrt{6}}{6 y(t) z(t)^2} d \star_t \left(\frac{y(t)^2 - z(t)^2}{y(t)^2 z(t)^2} (f^{1234} + f^{1256}) - \frac{2}{z(t)^2} f^{3456} \right) \\ &= -\frac{2 y(t)^4 - y(t)^2 z(t)^2 + z(t)^4}{6 y(t)^6 z(t)^8} (-f^{1234} - f^{1256}) - \frac{y(t)^2 - z(t)^2}{3 y(t)^4 z(t)^8} f^{3456}, \end{aligned}$$

that is

$$-\Delta_t \star_t \varphi(t) = -\frac{2 y(t)^4 - y(t)^2 z(t)^2 + z(t)^4}{6 y(t)^4 z(t)^6} (-e^{1234} - e^{1256}) - \frac{y(t)^2 - z(t)^2}{3 y(t)^4 z(t)^4} e^{3456}.$$

Therefore, (19) is the solution of the coflow (1) when $y = y(t)$ and $z = z(t)$ are the functions satisfying (20).

To prove that the solution (19) is ancient, we show that $\inf(I) = -\infty$ and $\sup(I) < +\infty$. To this aim, we first prove that the functions $y = y(t)$ and $z = z(t)$, solving (20), satisfy

$$(23) \quad y'(t) < 0, \quad t \in I,$$

$$(24) \quad z'(0) = 0,$$

$$(25) \quad \begin{cases} z'(t) < 0 & \text{or, equivalently,} \\ 1 < z(t) < y(t), & t \in I \cap (-\infty, 0), \end{cases}$$

$$(26) \quad \begin{cases} z'(t) > 0 & \text{or, equivalently,} \\ 0 < y(t) < 1 < z(t), & t \in I \cap (0, +\infty). \end{cases}$$

The properties (23) and (24) follow immediately from (20) since $y = y(t)$ and $z = z(t)$ are positive functions, and $y(0) = z(0) = 1$.

Using again that $y = y(t)$ and $z = z(t)$ are positive functions, the system (20) implies the following equivalence

$$(27) \quad z'(t) < 0 \quad \text{if and only if} \quad z(t) < y(t),$$

for $t \in I - \{0\}$. But the function $z = z(t)$ has a minimum at $t = 0$, since $z'(0) = 0$ and

$$z''(0) = -\frac{1}{6}y'(0) = \frac{1}{36} > 0.$$

Thus, according with (27), and using that $z(0) = 1$, to prove (25) it is sufficient to show that $z'(t) < 0$ if $t \in I \cap (-\infty, 0)$. For this, we proceed by contradiction. Suppose that there exists $s < 0$ such that

$$(28) \quad z'(t) < 0, \quad t \in (s, 0),$$

but

$$(29) \quad z'(s) = 0.$$

Then, taking into account (27), the inequality (28) implies that $y(t) > z(t)$, for every $t \in (s, 0)$, while the identity (29) is equivalent to $z(s) = y(s)$. It follows that

$$y'(s) = \lim_{t \rightarrow s^+} \frac{y(t) - y(s)}{t - s} \geq \lim_{t \rightarrow s^+} \frac{z(t) - z(s)}{t - s} = 0.$$

Hence, $y'(s) \geq 0$ contradicting the condition (23). Therefore, the inequality (25) is satisfied.

In order to prove (26), let us suppose that there exists $s > 0$ such that

$$z'(t) > 0, \quad t \in (0, s)$$

but $z'(s) = 0$. Then, (27) implies that $z(t) > y(t)$. Moreover, $t - s < 0$, and $z(s) = y(s)$ by (20). So,

$$y'(s) = \lim_{t \rightarrow s^-} \frac{y(t) - y(s)}{t - s} \geq \lim_{t \rightarrow s^+} \frac{z(t) - z(s)}{t - s} = 0,$$

in contradiction with (23).

Now we will prove, by contradiction, that $\inf(I) = -\infty$. Assume that $t_0 = \inf(I) > -\infty$. Then, (23) and (25) imply

$$1 = y(0) < y(t), \quad 1 = z(0) < z(t), \quad t \in (t_0, 0).$$

So, using (20), we get

$$\sup_{t_0 < t \leq 0} y(t) = 1 + \int_0^{t_0} y'(t) dt = 1 + \frac{1}{12} \int_{t_0}^0 \left(\frac{1}{z(t)^4 y(t)^5} + \frac{1}{z(t)^8 y(t)} \right) dt \leq 1 - \frac{t_0}{6},$$

since the functions $y = y(t)$ and $z = z(t)$ are decreasing for $t \leq 0$, and $y(0) = z(0) = 1$, so that $\left(\frac{1}{z(t)^4 y(t)^5} + \frac{1}{z(t)^8 y(t)} \right) \leq 2$, for $t \in (t_0, 0)$. Then, from (25) we obtain

$$z_0 = \sup_{t_0 < t \leq 0} z(t) \leq y_0 = \sup_{t_0 < t \leq 0} y(t) < +\infty.$$

Since y and z are decreasing for $t \leq 0$, we have

$$\lim_{t \rightarrow t_0^+} y(t) = y_0, \quad \lim_{t \rightarrow t_0^+} z(t) = z_0.$$

Then, by Cauchy Theorem, we can continue the solution below t_0 , in contradiction with the maximality of I .

To prove that $\sup(I) < +\infty$ suppose, on the contrary, that $\sup(I) = +\infty$. Using (20) and taking into account (26), we have

$$y''(t) = -\frac{1}{144} z^{-16} y^{-11} [z^2(z^4 + 2y^4)(5z^2 - 4y^2) + 9y^8] < 0, \quad t \in (0, +\infty).$$

This means that the function $y' = y'(t)$ is decreasing on $(0, +\infty)$. So, since $y'(0) = -\frac{1}{6}$,

$$y'(t) < y'(0) = -\frac{1}{6}, \quad t \in (0, +\infty).$$

Hence,

$$\lim_{t \rightarrow +\infty} y(t) = 1 + \int_0^{+\infty} y'(t) dt \leq 1 - \frac{1}{6} \int_0^{+\infty} dt = -\infty,$$

in contradiction with (26). Thus, $\sup(I) < +\infty$.

Regarding the metric $g_t = \sum_{i=1}^7 (f^i(t))^2$, one can check that, with respect to the basis $\{f_1(t), \dots, f_7(t)\}$ of \mathfrak{h} dual to the basis $\{f^1(t), \dots, f^7(t)\}$, the Ricci tensor $\text{Ric}(g_t)$

turns out to be

$$\text{Ric}(g_t) = -\frac{1}{12} y(t)^{-2} z(t)^{-4} \text{diag} \left(z(t)^{-4}, z(t)^{-4}, z(t)^{-4}, z(t)^{-4}, z(t)^{-4}, z(t)^{-4}, -y(t)^{-4} - 2 z(t)^{-4} \right).$$

Thus, if there is $t \in I$ such that g_t is a nilsoliton, that is the Ricci tensor $\text{Ric}(g_t)$ satisfies

$$\text{Ric}(g_t) = \lambda_t I + D_t,$$

then the derivation D_t of \mathfrak{h} , with respect to the basis $\{f_1(t), \dots, f_7(t)\}$, is given by a diagonal matrix. Now, using that D_t is a derivation of \mathfrak{h} , it is straightforward to verify that if g_t is a nilsoliton, then $z(t) = y(t)$. But, by (20), (25) and (26), we know that $z(t) = y(t)$ if and only if $t = 0$. So, among the coclosed G_2 forms $\varphi(t)$, the only one inducing a nilsoliton is $\varphi(0) = \varphi_2$. \square

5. EXPLICIT SOLUTIONS FOR THE MODIFIED LAPLACIAN COFLOW

We study the modified Laplacian coflow (2) for each of the coclosed G_2 forms defined in Proposition 1 on the 7-dimensional Heisenberg group. In particular, we prove that the solution of (2) for φ_1 is ancient only if the positive constant A , that appears in (2), take values in a certain open interval, while the solution of (2) for $-\varphi_1$ is ancient for any A . However, we prove that the solution of (2) for φ_2 is never ancient. We also show that the solution of (2) for φ_2 does not induce a nilsoliton for the whole time of existence.

Theorem 6. *The solution of the modified Laplacian coflow (2) for the coclosed G_2 form φ_1 , defined by (9), is given by*

$$(30) \quad \varphi(t) = \frac{1}{y(t)} \left(e^{127} + e^{347} + e^{567} \right) + y(t)^3 \left(e^{135} - e^{146} - e^{236} - e^{245} \right),$$

where the function $y = y(t)$ satisfies

$$(31) \quad \begin{cases} \frac{d}{dt} y(t) = \frac{2A\sqrt{6}y(t)^5 - 1}{12y(t)^9}, \\ y(0) = 1. \end{cases}$$

Moreover,

- i) if $0 < A < \frac{1}{2\sqrt{6}}$, then $t \in (-\infty, T)$, with $T = -\frac{1}{10A^2} \left(2\sqrt{6}A + \log(1 - 2\sqrt{6}A) \right) > 0$. Therefore, in this case, the solution (30) is ancient;
- ii) if $A \geq \frac{1}{2\sqrt{6}}$, then $t \in (-\infty, +\infty)$.

For any A , $\varphi(t)$ induces a nilsoliton for the whole time of existence.

Proof. By Cauchy Theorem, there exist a maximal open interval I , containing 0, and a smooth function $y : I \rightarrow (0, +\infty)$, which is the unique solution of (31).

To prove that (30) is the solution to the coflow (2) for φ_1 , we proceed as follows. As in the proof of Theorem 4, for each $t \in I$, we consider the basis $\{f^1(t), \dots, f^7(t)\}$ of left invariant 1-forms on H defined by

$$\begin{aligned} f^i &= f^i(t) e^i, \quad i = 1, \dots, 6, \\ f^7 &= f^7(t) e^7, \end{aligned}$$

where the function $y = y(t)$ satisfies now (31). Then, $f^i(0) = e^i$, for $i \in \{1, \dots, 7\}$, and the structure equations of H , with respect to the basis $\{f^1(t), \dots, f^7(t)\}$, are

$$(32) \quad df^i = 0, \quad 1 \leq i \leq 6, \quad df^7 = \frac{\sqrt{6}}{6} y^{-5}(t)(f^{12} + f^{34} + f^{56}).$$

Moreover, for any $t \in I$, the 3-form $\varphi(t)$ defined by (30) has the following expression

$$(33) \quad \varphi(t) = f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}.$$

So, $\varphi(0) = \varphi_1$ and, for any $t \in I$, the 3-form $\varphi(t)$ on H induces the metric g_t such that the coframe $\{f^1(t), \dots, f^7(t)\}$ of \mathfrak{h}^* is orthonormal. Denote by \star_t the Hodge star operator determined by g_t . Using (4), (5) and (32), we have $d \star_t \varphi(t) = 0$, where $\star_t \varphi(t)$ is given by

$$\star_t \varphi(t) = f^{1234} + f^{1256} + f^{1367} + f^{1457} + f^{2357} - f^{2467} + f^{3456}.$$

Thus, in terms of the coframe $\{e^1, \dots, e^7\}$ of \mathfrak{h}^* , the 4-form $\star_t \varphi(t)$ has the following expression

$$\star_t \varphi(t) = y(t)^4 (e^{1234} + e^{1256} + e^{3456}) + e^{1367} + e^{1457} + e^{2357} - e^{2467}.$$

This implies

$$\frac{d}{dt} \star_t \varphi(t) = 4 y(t)^3 y'(t) (e^{1234} + e^{1256} + e^{3456}),$$

that is

$$(34) \quad \frac{d}{dt} \star_t \varphi(t) = \frac{2A\sqrt{6}y(t)^5 - 1}{3y(t)^6} (e^{1234} + e^{1256} + e^{3456}),$$

since the function $y = y(t)$ satisfies (31).

On the other hand, by (11) we know that the torsion forms $\tau_i(t)$ ($i = 0, 1, 2, 3$) of $\varphi(t)$ are such that $\tau_1(t) = 0 = \tau_2(t)$ since $d(\star_t \varphi(t)) = 0$. Then, from (32), (33) and (11), we have

$$(35) \quad d\varphi(t) = \frac{\sqrt{6}}{3y(t)^5} (f^{1234} + f^{1256} + f^{3456}) = \tau_0(t) \star_t \varphi(t) + \star_t \tau_3(t),$$

where

$$\tau_3(t) = \frac{\sqrt{6}}{7y(t)^5} (-f^{135} + f^{146} + f^{236} + f^{245}) + \frac{4\sqrt{6}}{21y(t)^5} (f^{127} + f^{347} + f^{567}),$$

$$\star_t \tau_3(t) = \frac{\sqrt{6}}{7y(t)^5}(-f^{1367} - f^{1457} - f^{2357} + f^{2467}) + \frac{4\sqrt{6}}{21y(t)^5}(f^{1234} + f^{1256} + f^{3456}),$$

and

$$\tau_0(t) = \frac{\sqrt{6}}{7y(t)^5}.$$

So, according with the first equality of (35),

$$\begin{aligned} \Delta_t \star_t \varphi(t) + 2d\left((A - \frac{7}{4}\tau_0)\varphi(t)\right) &= d \star_t d(\varphi(t)) + 2\left(A - \frac{7}{4}\tau_0\right)d\varphi(t) \\ &= \frac{2A\sqrt{6}y(t)^5-1}{3y(t)^{10}}(f^{1234} + f^{1256} + f^{3456}), \end{aligned}$$

that is

$$\Delta_t \star_t \varphi(t) + 2d\left((A - \frac{7}{4}\tau_0)\varphi(t)\right) = \frac{2A\sqrt{6}y(t)^5-1}{3y(t)^6}(e^{1234} + e^{1256} + e^{3456}).$$

The last equality, together with (12) and (34), show that (30) solves the modified Laplacian coflow (2) for φ_1 .

In order to show that the solution $\varphi(t)$, given by (30), is ancient, we analyse the behaviour of the function $y = y(t)$ according with the values of the positive constant A . If $A = \frac{1}{2\sqrt{6}}$, then (31) implies that $y'(t) = 0$. Thus, $y(t) \equiv 1$, for all $t \in (-\infty, +\infty)$, since $y(0) = 1$. Assume $A \neq \frac{1}{2\sqrt{6}}$. In this case, we first observe that $y = y(t)$ is monotone. Indeed, the constant function $\hat{y}(t) \equiv (2\sqrt{6}A)^{-1/5}$ satisfies the differential equation that appears in (31), which is autonomous, and so their solutions are monotone functions. Hence, the solution $y = y(t)$ of the system (31) is monotone and it must satisfy either $y(t) > \hat{y}(t)$ or $y(t) < \hat{y}(t)$ for any $t \in I$, according to the value of A and the fact $y(0) = 1$, that is if $2\sqrt{6}A < 1$, then $y(t) < \hat{y}(t)$, but if $2\sqrt{6}A > 1$, then $y(t) > \hat{y}(t)$.

Now, we rewrite the differential equation that appears in (31) as

$$\left(\frac{\sqrt{6}}{A}y(t)^4 + \frac{\sqrt{6}}{A} \frac{y(t)^4}{2\sqrt{6}Ay(t)^5 - 1} \right) y'(t) = 1.$$

Integrating this equation from 0 to t , we have

$$(36) \quad t = \frac{\sqrt{6}}{5A}(y(t)^5 - 1) + \frac{1}{10A^2} \log \left| \frac{1 - 2\sqrt{6}Ay(t)^5}{1 - 2\sqrt{6}A} \right|.$$

Therefore, taking into account (36), if $2\sqrt{6}A < 1$, then $y = y(t)$ decreases from $(2\sqrt{6}A)^{-1/5}$ to 0 as t goes from $-\infty$ to $-\frac{2A\sqrt{6} + \log(1-2\sqrt{6}A)}{10A^2}$. Otherwise, if $2\sqrt{6}A > 1$, then $y = y(t)$ increases from $(2\sqrt{6}A)^{-1/5}$ to $+\infty$ as t goes from $-\infty$ to $+\infty$. In particular, we have that the definition interval I of the function $y = y(t)$ is

$$I = (-\infty, -\frac{2\sqrt{6}A + \log(1-2\sqrt{6}A)}{10A^2}), \quad \text{if } A < \frac{1}{2\sqrt{6}},$$

and

$$I = (-\infty, +\infty), \quad \text{if } A \geq \frac{1}{2\sqrt{6}}.$$

To complete the proof, we show that the metric g_t determined by $\varphi(t)$ is a nilsoliton. We know that $g_t = \sum_{i=1}^7 (f^i(t))^2$. Then, as in Theorem 4, the Ricci tensor $\text{Ric}(g_t)$, with respect to the basis $\{f_1(t), \dots, f_7(t)\}$ of \mathfrak{h} , dual to the basis $\{f^1(t), \dots, f^7(t)\}$ of \mathfrak{h}^* , is given by

$$\text{Ric}(g_t) = y(t)^{-10} \text{diag} \left(-\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{4} \right),$$

where $y = y(t)$ is now the function satisfying (31). Thus, as in Theorem 4,

$$\text{Ric}(g_t) = \lambda_t I + D_t,$$

with $\lambda_t = -\frac{1}{12} y(t)^{-10}$, and

$$D_t = \text{diag} \left(0, 0, 0, 0, 0, 0, \frac{1}{3} y(t)^{-10} \right),$$

which is a derivation of \mathfrak{h} , for every fixed t . Hence, g_t is a nilsoliton on H , for every t . \square

Remark 4. In a similar way as in the proof of Theorem 4, one can check that the Riemannian curvature $R(g_t)$ of the metric g_t induced by (30) is such that

$$\|R(g_t)\|_{g_t}^2 = \frac{23}{48} y(t)^{-20},$$

and so, in the case iii) (corresponding to $A > \frac{1}{2\sqrt{6}}$) $\lim_{t \rightarrow +\infty} R(g_t) = 0$.

In the following theorem we study the modified Laplacian coflow (2) when the initial coclosed G_2 form on the 7-dimensional Heisenberg group is equal to $-\varphi_1$, where φ_1 is defined by (9).

Theorem 7. *The solution of the modified Laplacian coflow (2) with initial coclosed G_2 form $-\varphi_1$ is ancient and it is given by*

$$(37) \quad \varphi(t) = -\frac{1}{y(t)} (e^{127} + e^{347} + e^{567}) - y(t)^3 (e^{135} - e^{146} - e^{236} - e^{245}),$$

where $t \in (-\infty, T)$, with $T = \frac{\sqrt{6}}{5A} \left(1 - (2A\sqrt{6})^{-1} \log(2A\sqrt{6} + 1) \right)$, and the function $y = y(t)$ satisfies

$$(38) \quad \begin{cases} \frac{d}{dt} y(t) = -\frac{2A\sqrt{6}y(t)^5 + 1}{12y(t)^9}, \\ y(0) = 1. \end{cases}$$

The underlying metrics g_t of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in H as t goes

to $-\infty$. Moreover, the metric g_t induced by $\varphi(t)$ is a nilsoliton for the whole time of existence.

Proof. By Cauchy's Theorem there exist a maximal open interval I , containing 0, and a smooth function $y : I \rightarrow (0, +\infty)$, which is the unique solution of (38).

To prove that (37) is the solution of the coflow (2) for $-\varphi_1$, we proceed as follows. As in the proof of Theorem 6, for each $t \in I$, we consider the basis $\{f^1(t), \dots, f^7(t)\}$ of left invariant 1-forms on H defined by

$$\begin{aligned} f^i &= f^i(t) = y(t) e^i, \quad i = 1, \dots, 6 \\ f^7 &= f^7(t) = y(t)^{-3} e^7, \end{aligned}$$

where the function $y = y(t)$ satisfies now (38). Then, $f^i(0) = e^i$, for $i \in \{1, \dots, 7\}$, and the structure equations of H , with respect to the basis $\{f^1(t), \dots, f^7(t)\}$, are

$$(39) \quad df^i = 0, \quad 1 \leq i \leq 6, \quad df^7 = \frac{\sqrt{6}}{6} y(t)^{-5} (f^{12} + f^{34} + f^{56}).$$

Now, for any $t \in I$, the 3-form $\varphi(t)$ defined by (37) has the following expression

$$(40) \quad \varphi(t) = -(f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}).$$

So, $\varphi(0) = -\varphi_1$ and, for any $t \in I$, the metric g_t induced by $\varphi(t)$ is such that the coframe $\{f^1(t), \dots, f^7(t)\}$ of \mathfrak{h}^* is orthonormal. Denote by \star_t the Hodge star operator determined by g_t . Using (39), we have $d\star_t \varphi(t) = 0$, where $\star_t \varphi(t)$ is given by

$$\star_t \varphi(t) = f^{1234} + f^{1256} + f^{1367} + f^{1457} + f^{2357} - f^{2467} + f^{3456}.$$

Then, in terms of the coframe $\{e^1, \dots, e^7\}$ of \mathfrak{h}^* , the 4-form $\star_t \varphi(t)$ has the following expression

$$\star_t \varphi(t) = y(t)^4 (e^{1234} + e^{1256} + e^{3456}) + e^{1367} + e^{1457} + e^{2357} - e^{2467}.$$

Therefore,

$$\frac{d}{dt} \star_t \varphi(t) = 4y(t)^3 y'(t) (e^{1234} + e^{1256} + e^{3456}),$$

that is,

$$(41) \quad \frac{d}{dt} \star_t \varphi(t) = -\frac{2A\sqrt{6}y(t)^5 + 1}{3y(t)^6} (e^{1234} + e^{1256} + e^{3456}),$$

since the function $y = y(t)$ satisfies (38).

On the other hand, by (11) we know that the torsion forms $\tau_i(t)$ ($i = 0, 1, 2, 3$) of $\varphi(t)$ are such that $\tau_1(t) = 0 = \tau_2(t)$ since $d(\star_t \varphi(t)) = 0$. Then, from (39), (40) and using again (11), we have

$$(42) \quad d\varphi(t) = -\frac{\sqrt{6}}{3y(t)^5} (f^{1234} + f^{1256} + f^{3456}) = \tau_0(t) \star_t \varphi(t) + \star_t \tau_3(t),$$

where

$$\tau_3(t) = \frac{\sqrt{6}}{7y(t)^5}(-f^{135} + f^{146} + f^{236} + f^{245}) + \frac{4\sqrt{6}}{21y(t)^5}(f^{127} + f^{347} + f^{567}),$$

$$\star_t \tau_3(t) = \frac{\sqrt{6}}{7y(t)^5}(-f^{1367} - f^{1457} - f^{2357} + f^{2467}) + \frac{4\sqrt{6}}{21y(t)^5}(f^{1234} + f^{1256} + f^{3456}),$$

and

$$\tau_0(t) = -\frac{\sqrt{6}}{7y(t)^5}.$$

Then, according with the first equality of (42),

$$\begin{aligned} \Delta_t \star_t \varphi(t) + 2d\left((A - \frac{7}{4}\tau_0)\varphi(t)\right) &= d \star_t d(\varphi(t)) + 2(A - \frac{7}{4}\tau_0)d\varphi(t) \\ &= -\frac{2A\sqrt{6}y(t)^5+1}{3y(t)^{10}}(f^{1234} + f^{1256} + f^{3456}), \end{aligned}$$

or, equivalently,

$$\Delta_t \star_t \varphi(t) + 2d\left((A - \frac{7}{4}\tau_0)\varphi(t)\right) = -\frac{2A\sqrt{6}y(t)^5+1}{3y(t)^6}\left(e^{e^{1234}+1256} + e^{3456}\right).$$

The last equality, together with (12) and (41), show that (37) solves the modified Laplacian flow (2) for $-\varphi_1$.

To show that the solution $\varphi(t)$, given by (37), is ancient, we study the behaviour of the function $y = y(t)$. To this end, we rewrite the differential equation that appears in (38) as

$$-\frac{12y(t)^9}{2A\sqrt{6}y(t)^5+1}y' = 1.$$

Integrating this equation from 0 to t we obtain

$$(43) \quad \frac{\sqrt{6}}{5A}(1 - y^5(t)) + \frac{1}{10A^2}\log\left(\frac{2A\sqrt{6}y^5(t)+1}{2A\sqrt{6}+1}\right) = t.$$

Clearly $y'(t) < 0$ since the function $y = y(t)$ satisfies the differential equation that appears in (38). Then, (43) implies that the function $y = y(t)$ decreases from $+\infty$ to 0 as t goes from $-\infty$ to $\frac{\sqrt{6}}{5A}\left(1 - \frac{1}{2A\sqrt{6}}\log(2A\sqrt{6}+1)\right)$.

To study the behaviour of the underlying metric g_t of the solution (37) for $t \rightarrow -\infty$, we proceed in a similar way as in the proof of Theorem 4, and we have

$$\|R(g_t)\|_{g_t}^2 = \frac{23}{48}y(t)^{-20}.$$

So $\lim_{t \rightarrow -\infty} R(g_t) = 0$.

To complete the proof, we show that the metric g_t associated to the solution $\varphi(t)$ is a nilsoliton for any $t \in I$. We know that $g_t = \sum_{i=1}^7 (f^i(t))^2$. Thus, as in Theorem

4, the Ricci tensor $\text{Ric}(g_t)$, with respect to the basis $\{f_1(t), \dots, f_7(t)\}$, is given by

$$\text{Ric}(g_t) = y(t)^{-10} \text{diag} \left(-\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{4} \right) = \lambda_t \text{Id} + D_t,$$

where $\lambda_t = -\frac{1}{12}y(t)^{-10}$, and

$$D_t = \text{diag} \left(0, 0, 0, 0, 0, 0, \frac{1}{3}y(t)^{-10} \right),$$

which is a derivation of \mathfrak{h} , for every fixed t . Therefore, g_t is a nilsoliton on H , for every $t \in I$. \square

Concerning the modified Laplacian coflow (2) for the coclosed G_2 form φ_2 on the 7-dimensional Heisenberg group H we have the following.

Theorem 8. *The solution of the modified Laplacian coflow (2) with initial coclosed G_2 -structure φ_2 is defined on a bounded interval, and it is given by*

$$(44) \quad \varphi(t) = \frac{y(t)}{z(t)^2} e^{127} - y(t)^{-1} (e^{347} + e^{567}) + y(t)z(t)^2 (e^{135} - e^{146} + e^{236} + e^{245}),$$

where the functions $y = y(t)$ and $z = z(t)$ satisfy

$$(45) \quad \begin{cases} \frac{d}{dt}y(t) = \frac{2A\sqrt{6}y(t)z(t)^6 + 2z(t)^2 + y(t)^2}{12y(t)^3z(t)^8}, & \frac{d}{dt}z(t) = -\frac{2A\sqrt{6}y(t)z(t)^4 + 1}{12y(t)^2z(t)^7}, \\ y(0) = 1, & z(0) = 1. \end{cases}$$

Moreover, $\varphi(t)$ induces a nilsoliton only for $t = 0$.

Proof. By Cauchy's Theorem, there exist a maximal open interval I , containing 0, and two smooth functions $y, z : I \rightarrow (0, +\infty)$, which are the unique solution of (45).

We first prove that (44) is the solution of the coflow (2) for φ_2 . As in the proof of Theorem 5, for each $t \in I$, we consider the basis $\{f^1(t), \dots, f^7(t)\}$ of left invariant 1-forms on H defined by

$$\begin{aligned} f^i &= f^i(t) = y(t) e^i, & i &= 1, 2, \\ f^i &= f^i(t) = z(t) e^i, & i &= 3, \dots, 6, \\ f^7 &= f^7(t) = y(t)^{-1} z(t)^{-2} e^7, \end{aligned}$$

where the functions $y = y(t)$ and $z = z(t)$ satisfy now (45). Then, $f^i(0) = e^i$, for $i \in \{1, \dots, 7\}$, and the structure equations of H , with respect to the basis $\{f^1(t), \dots, f^7(t)\}$, are

$$(46) \quad \begin{aligned} df^i &= 0, \quad 1 \leq i \leq 6, \\ df^7 &= \frac{\sqrt{6}}{6} y(t)^{-1} z(t)^{-2} \left(y(t)^{-2} f^{12} + z(t)^{-2} f^{34} + z(t)^{-2} f^{56} \right). \end{aligned}$$

Moreover, for any $t \in I$, the 3-form $\varphi(t)$ defined by (44) has the following expression

$$(47) \quad \varphi(t) = f^{127} - f^{347} - f^{567} + f^{135} - f^{146} + f^{236} + f^{245}.$$

So $\varphi(0) = \varphi_2$ and, for any $t \in I$, the 3-form $\varphi(t)$ on H induces the metric g_t such that $\{f^1(t), \dots, f^7(t)\}$ of \mathfrak{h}^* is an orthonormal basis of \mathfrak{h}^* . Denote by \star_t the Hodge operator determined by g_t . Using (4), (5) and (46), we have $d \star_t \varphi(t) = 0$, where $\star_t \varphi(t)$ is given by

$$\star_t \varphi(t) = -f^{1234} - f^{1256} - f^{1367} - f^{1457} + f^{2357} - f^{2467} + f^{3456}.$$

Thus, in terms of the coframe $\{e^1, \dots, e^7\}$ of \mathfrak{h}^* , the 4-form $\star_t \varphi(t)$ has the following expression

$$\star_t \varphi(t) = y(t)^2 z(t)^2 (-e^{1234} - e^{1256}) - e^{1367} - e^{1457} + e^{2357} - e^{2467} + z(t)^4 e^{3456}.$$

Therefore,

$$\frac{d}{dt} (\star_t \varphi(t)) = 2 \left(y(t) z(t)^2 y'(t) + y(t)^2 z(t) z'(t) \right) (-e^{1234} - e^{1256}) + 4 z(t)^3 z'(t) e^{3456},$$

that is

$$(48) \quad \begin{aligned} \frac{d}{dt} \star_t \varphi(t) &= \frac{A\sqrt{6}(y(t)^3 z(t)^2 - y(t) z(t)^4) - 1}{3 y(t)^2 z(t)^4} (e^{1234} + e^{1256}) \\ &\quad - \frac{2A\sqrt{6} y(t) z(t)^4 + 1}{3 y(t)^2 z(t)^4} e^{3456}, \end{aligned}$$

since the functions $y = y(t)$ and $z = z(t)$ satisfy (45).

On the other hand, let us consider the torsion forms $\tau_i(t)$ ($i = 0, 1, 2, 3$) of $\varphi(t)$. By (11), $\tau_1(t) = 0 = \tau_2(t)$ since $d(\star_t \varphi(t)) = 0$. Then, from (46), (47) and using again (11), we have

$$(49) \quad \begin{aligned} d\varphi(t) &= \frac{\sqrt{6}}{6} y(t)^{-1} z(t)^{-2} \left((z(t)^{-2} - y(t)^{-2}) (f^{1234} + f^{1256}) - 2z(t)^{-2} f^{3456} \right) \\ &= \tau_0(t) \star_t \varphi(t) + \star_t \tau_3(t), \end{aligned}$$

where

$$\begin{aligned} \tau_3(t) &= -\frac{\sqrt{6}(5y(t)^2 + z(t)^2)}{21 y(t)^3 z(t)^4} f^{127} + \frac{\sqrt{6}(3y(t)^2 - 5z(t)^2)}{42 y(t)^3 z(t)^4} (f^{347} + f^{567}) \\ &\quad + \frac{\sqrt{6}(2y(t)^2 - z(t)^2)}{21 y(t)^3 z(t)^4} (f^{135} - f^{146} + f^{236} + f^{245}), \\ \star_t \tau_3(t) &= -\frac{\sqrt{6}(5y(t)^2 + z(t)^2)}{21 y(t)^3 z(t)^4} f^{3456} + \frac{\sqrt{6}(3y(t)^2 - 5z(t)^2)}{42 y(t)^3 z(t)^4} (f^{1234} + f^{1256}) \\ &\quad + \frac{\sqrt{6}(2y(t)^2 - z(t)^2)}{21 y(t)^3 z(t)^4} (-f^{1367} - f^{1457} + f^{2357} - f^{2467}), \end{aligned}$$

and

$$\tau_0(t) = -\frac{\sqrt{6}}{21y(t)^3z(t)^4}(2y(t)^2 - z(t)^2).$$

Then, according with the first equality of (49),

$$\begin{aligned} \Delta_t \star_t \varphi(t) + 2d\left((A - \frac{7}{4}\tau_0)\varphi(t)\right) &= d \star_t d(\varphi(t)) + 2(A - \frac{7}{4}\tau_0)d\varphi(t) \\ &= \frac{A\sqrt{6}\left(\frac{y(t)^3z(t)^2 - y(t)z(t)^4}{3y(t)^4z(t)^6} - 1\right)}{f^{1234} + f^{1256}} \\ &\quad - \frac{2A\sqrt{6}y(t)z(t)^4 + 1}{3y(t)^2z(t)^8}f^{3456}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \Delta_t \star_t \varphi(t) + 2d\left((A - \frac{7}{4}\tau_0)\varphi(t)\right) &= \frac{A\sqrt{6}\left(\frac{y(t)^3z(t)^2 - y(t)z(t)^4}{3y(t)^2z(t)^4} - 1\right)}{e^{1234} + e^{1256}} \\ &\quad - \frac{2A\sqrt{6}y(t)z(t)^4 + 1}{3y(t)^2z(t)^4}e^{3456}. \end{aligned}$$

The last equality, together with (12) and (48), show that (44) solves the modified Laplacian flow (2) for φ_2 .

To prove that (44) is defined on a bounded interval, we will show that $t_+ = \sup(I) < +\infty$ and $t_- = \inf(I) > -\infty$. On the one hand, we know that the functions $y = y(t)$ and $z = z(t)$ are positive. Then, the system (45) implies that $z'(t) < 0 < y'(t)$, for any $t \in I$. Therefore, the function $z = z(t)$ is decreasing, and $y = y(t)$ is increasing. Thus, there exist

$$\lim_{t \rightarrow t_-} y(t) = y_- \in [0, 1) \quad \text{and} \quad \lim_{t \rightarrow t_+} z(t) = z_+ \in [0, 1).$$

Now, using (45), it is straightforward to verify that the function $z'' = z''(t)$ satisfies

$$z'' = -\frac{1}{144y^6z^{15}}\left(24A^2(3y^4z^8 - y^2z^{10}) + 2A\sqrt{6}(9y^3z^4 - 4yz^6) + 5y^2 - 4z^2\right),$$

for any $t \in I$. Note that in the last equality, the functions $(3y^4z^8 - y^2z^{10}) = y^2z^8(3y^2 - z^2)$, $(9y^3z^4 - 4yz^6) = yz^4(9y^2 - 4z^2)$ and $(5y^2 - 4z^2)$ are positive functions in $(0, t_+)$. Indeed, their values at $t = 0$ are positive, and $z = z(t)$ decreases while $y = y(t)$ increases in $(0, t_+)$. Therefore, $z''(t) < 0$, for $t \in (0, t_+)$. Thus, $z'(t) < z'(0) < 0$, for any $t \in (0, t_+)$. Now, we choose a sequence $\{t_n\} \subset I$ of positive times converging to t_+ . Then,

$$z(t_n) - 1 = \int_0^{t_n} z'(t) dt < \int_0^{t_n} z'(0) dt < z'(0) t_n.$$

So, $t_n < \frac{z(t_n)-1}{z'(0)}$ and, consequently, $t_+ \leq \frac{z_+-1}{z'(0)} < +\infty$.

Using again (45), we have

$$\begin{aligned} (50) \quad -144y^7z^{16}y'' &= 48A^2(z^{12}y^2 - z^{10}y^4) + 2A\sqrt{6}(10z^8y - 11z^6y^3 - 8z^4y^5) \\ &\quad + 12z^4 - 4z^2y^2 - 7y^4. \end{aligned}$$

Then, it is possible to show that $y''(t) < 0$ in some neighbourhood of t_- . Indeed, the functions $z^{12}y^2 - z^{10}y^4$ and

$$(12z^4 - 4z^2y^2 - 7y^4) = 4z^2(z^2 - y^2) + (8z^4 - 7y^4)$$

are both positive on $(t_-, 0)$, since the functions $z^2 - y^2$ and $8z^4 - 7y^4$ are both decreasing. Moreover, the solution is maximal for t going to t_- . Therefore, the limits $\lim_{t \rightarrow t_-} z(t) = z_-$ and $\lim_{t \rightarrow t_-} y(t) = y_-$ cannot be both finite and different from zero, otherwise we can restart the flow. As a consequence, since $y'(t) > 0$ and $z'(t) < 0$, for any $t \in I$, we get that either $z_- < +\infty$ (and consequently $y_- = 0$) or $z_- = +\infty$.

In the first case, the leading term (as polynomial in z) of the right side of (50) is $12z^4$, so it must be positive in a neighbourhood of t_- . On the other hand $-144y^7z^{16} < 0$, so $y''(t) < 0$ in some neighbourhood of t_- . In the other case (i.e. when $z_- = +\infty$),

$$\lim_{t \rightarrow t_-} (10z^8 - 11z^6y^2 - 8z^4y^4) = +\infty$$

since $z_- = +\infty$ and y is bounded. Therefore $y(10z^8 - 11z^6y^2 - 8z^4y^4)$ is positive in some neighbourhood of t_- . Hence, in both cases, it follows that $y'' < 0$ for $t \in (\bar{t}, t_-)$, for some $\bar{t} \in (t_-, 0)$, i.e. that $y'(t) > y'(\bar{t})$, for $t \in (t_-, \bar{t})$. Now, we choose a sequence of negative times $\{t_n\} \subset (t_-, \bar{t})$ converging to t_- . Then,

$$y(\bar{t}) - y(t_n) = \int_{t_n}^{\bar{t}} y'(t) dt > \int_{t_n}^{\bar{t}} y'(\bar{t}) dt = (\bar{t} - t_n) y'(\bar{t}).$$

It follows that $t_n > \frac{y(t_n) - y(\bar{t})}{y'(\bar{t})} + \bar{t}$. So, $t_- \geq \frac{y_- - y(\bar{t})}{y'(\bar{t})} + \bar{t} > -\infty$.

Finally, let us consider the metric $g_t = \sum_{i=1}^7 (f^i(t))^2$ induced by (44). Let $\{f_1(t), \dots, f_7(t)\}$ be the basis of \mathfrak{h} dual to the basis $\{f^1(t), \dots, f^7(t)\}$. Then, as in the proof of Theorem 5, the Ricci tensor $\text{Ric}(g_t)$, with respect to the basis $\{f_1(t), \dots, f_7(t)\}$, turns out to be

$$\text{Ric}(g_t) = -\frac{1}{12} y(t)^{-2} z(t)^{-4} \text{diag} \left(z(t)^{-4}, z(t)^{-4}, z(t)^{-4}, z(t)^{-4}, z(t)^{-4}, z(t)^{-4}, -y(t)^{-4} - 2z(t)^{-4} \right).$$

Thus, if there exist $t \in I$ such that g_t is a nilsoliton, that is

$$\text{Ric}(g_t) = \lambda_t I + D_t,$$

then the derivation D_t of \mathfrak{h} , with respect to the basis $\{f_1(t), \dots, f_7(t)\}$, is given by a diagonal matrix. Now, using that D_t is a derivation of \mathfrak{h} , it is straightforward to verify that g_t is a nilsoliton if and only if $y(t) = z(t)$. We know that $z(t) = y(t)$ if and only if $t = 0$. \square

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